Exercise 102

(a) Find the linear approximation to $f(x) = \sqrt{25 - x^2}$ near 3.

(b) Illustrate part (a) by graphing f and the linear approximation.

(c) For what values of x is the linear approximation accurate to within 0.1?

Solution

Part (a)

Plug in 3 to the function to determine the corresponding y-coordinate.

$$f(3) = \sqrt{25 - 3^2} = 4$$

This means the linear approximation touches $f(x)$ at the point $(3, 4)$. Now find the slope here by taking the derivative of $f(x)$,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\sqrt{25 - x^2} \\ &= \frac{1}{2}(25 - x^2)^{-1/2} \cdot \frac{d}{dx}(25 - x^2) \\ &= \frac{1}{2}(25 - x^2)^{-1/2} \cdot (-2x) \\ &= -\frac{x}{\sqrt{25 - x^2}}, \end{aligned}$$

and setting $x = 3$.

$$f'(3) = -\frac{3}{\sqrt{25 - 3^2}} = -\frac{3}{4}$$

Use the point-slope formula to obtain the equation of the line with this slope that goes through $(3, 4)$.

$$y - f(3) = f'(3)(x - 3)$$

$$y - 4 = -\frac{3}{4}(x - 3)$$

$$y - 4 = -\frac{3}{4}x + \frac{9}{4}$$

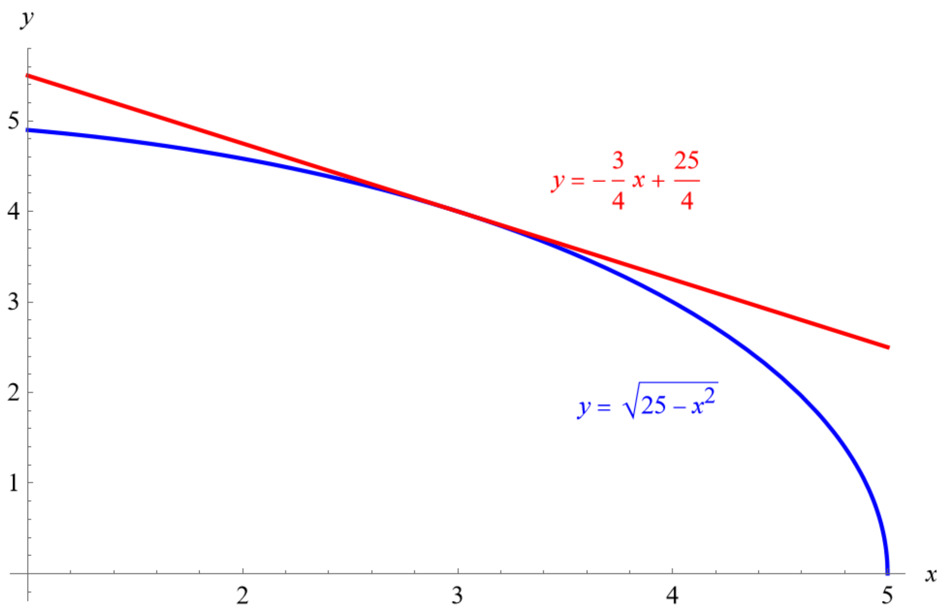
$$y = -\frac{3}{4}x + \frac{25}{4}$$

Therefore, the linear approximation to $f(x)$ at 3 is

$$L(x) = -\frac{3}{4}x + \frac{25}{4}.$$

Part (b)

Below is a graph of the function and its linear approximation at 3.



Part (c)

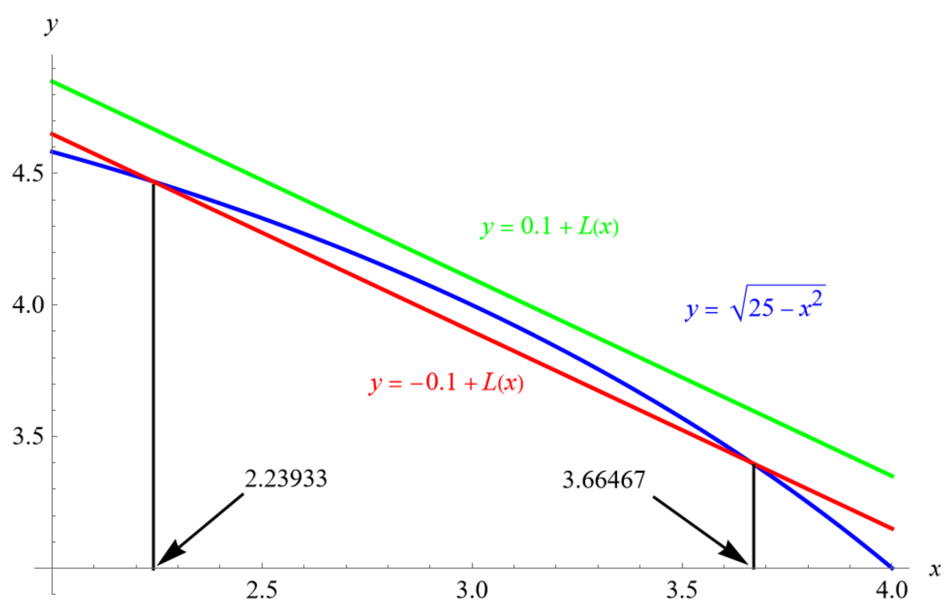
In order for the linear approximation to be accurate to within 0.1, the following inequality needs to be solved for x.

$$|f(x) - L(x)| < 0.1$$

$$-0.1 < f(x) - L(x) < 0.1$$

$$-0.1 + L(x) < f(x) < 0.1 + L(x)$$

As long as the curve stays between the lines defined by $y = -0.1 + L(x)$ and $y = 0.1 + L(x)$, the linear approximation will be accurate to within 0.1.



This occurs for

$$2.23933 < x < 3.66467.$$